Representations of $U_q(sl_2)$

Overview: In this section, we only focus on fin. dim simple U-mod and the center of $U_q(sl_2)$. For rep theory:

1. q is not a root of unity, $U_q(sl_2)/_k$ behaves like $U(sl_2)/_{\text{char } 0}$.

   where char $k \neq 2$.

2. q is a primitive l-th root of unity, with l odd and $l \geq 3$,

   $U_q(sl_2)/_{\text{alg closed } k}$ behaves like $U(sl_2)/_{\text{prime char}}$

For the center:

1. If q is not a root of unity, $C(U)$ is generated by C as a k-alg.
2. If q is a primitive l-th root of unity with l odd and $l \geq 3$.

   $C(U)$ is generated by E^l, F^l, and its intersection with U_0.

   $(C(U) = \langle E^l, F^l, C(U) \cap U_0 \rangle)$

Representation theory:

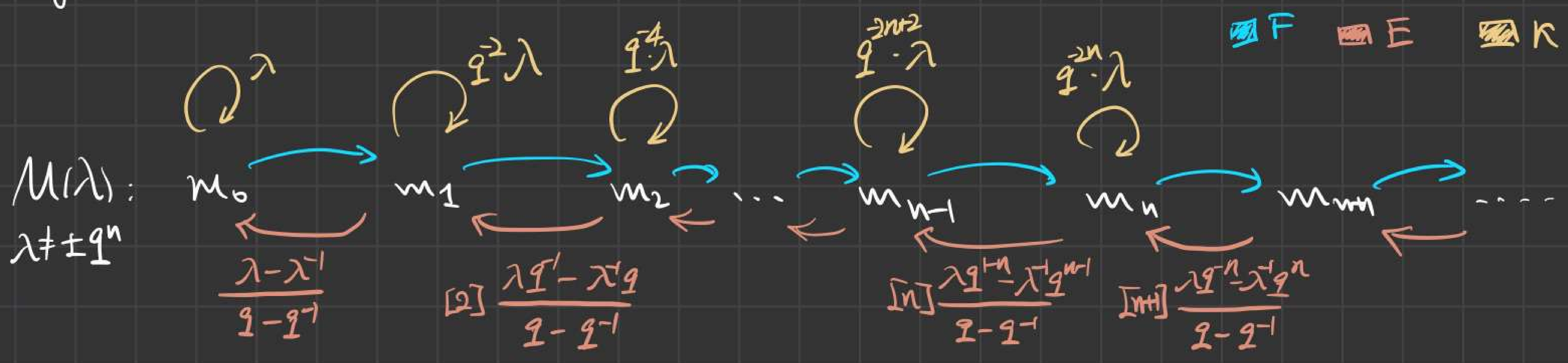
Let M be a U-mod, $\quad EM_\lambda \subset M_{q^2\lambda}$, $FM_\lambda \subset M_{q^{-2}\lambda}$

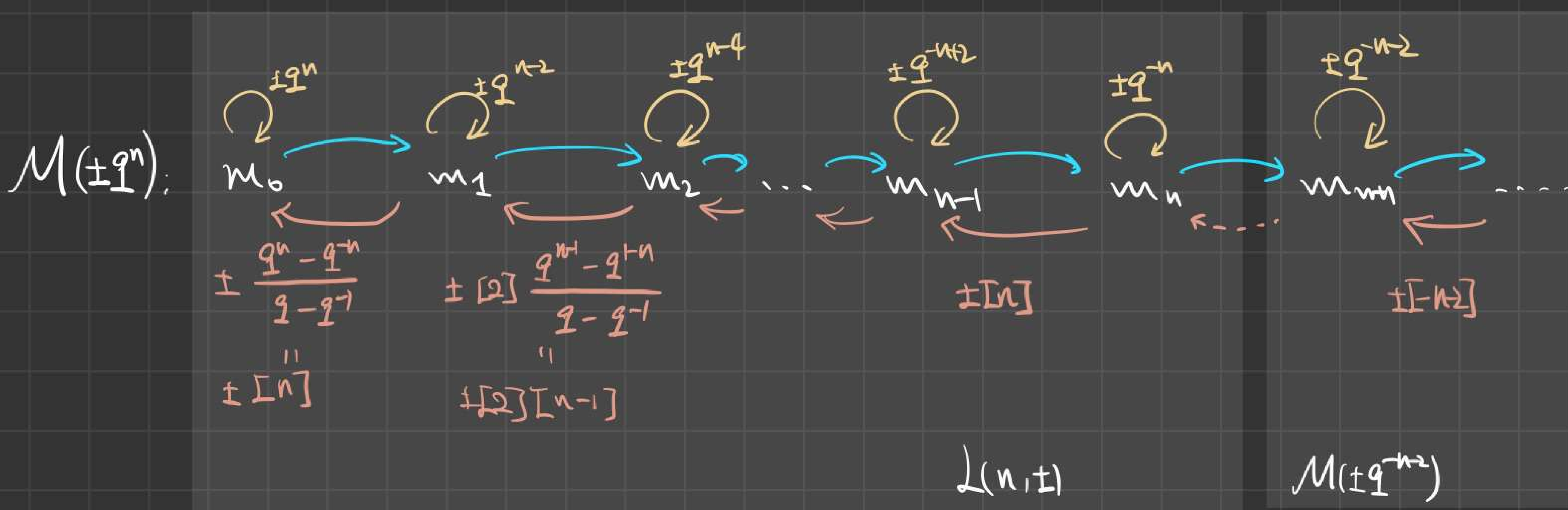
1. q is not a root of unity. Let M be a fin. dim U-mod. char $k \neq 2$

Then ① $\exists\, r, s \in \mathbb{N}$ s.t. $E^r M = F^s M = 0$. $\quad$ (E, F act nilpotently)

② $M = \bigoplus_{a \in \mathbb{Z}} M_{\pm q^a}$. (We can find a zero polynomial of K: $\prod_{j=-(s-1)}^{s-1} (K^2 - q^{2j})$)

③ If M is a (fin. dim) simple $\mathcal{U}$-mod, then $M \simeq L(n,\pm) \simeq \dfrac{M(\pm q^n)}{M(\pm q^{-n-2})}$ for some $n \in \mathbb{N}$.

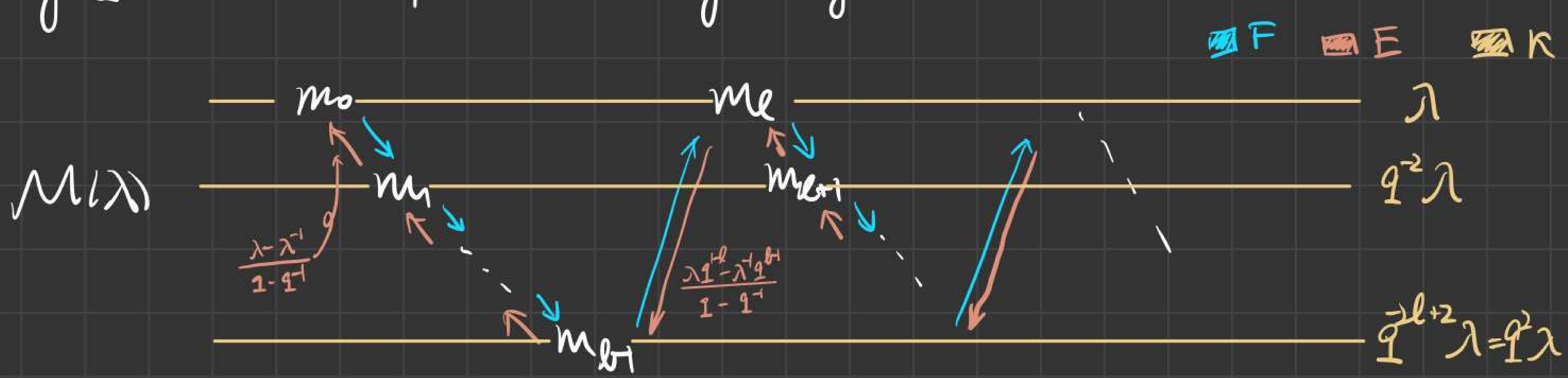


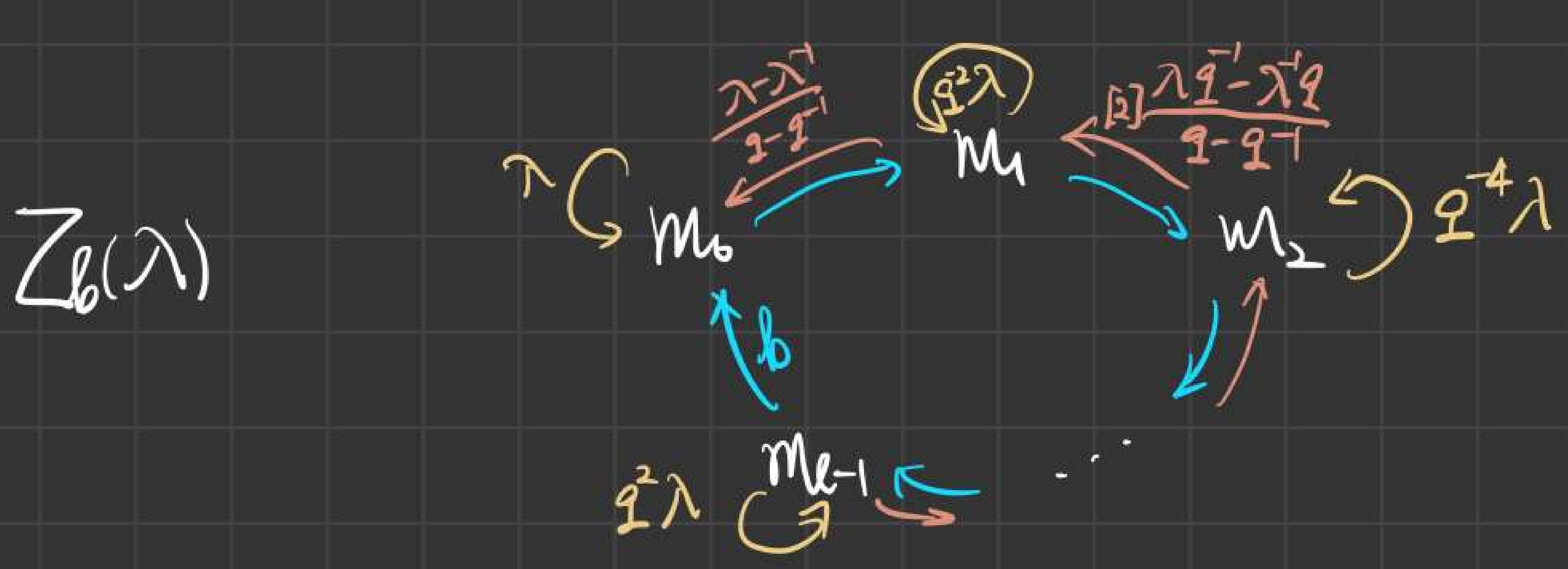


$0 \to M(\pm q^{-n-2}) \longrightarrow M(\pm q^n) \longrightarrow L(n,\pm) \to 0$ NOT Split SES!

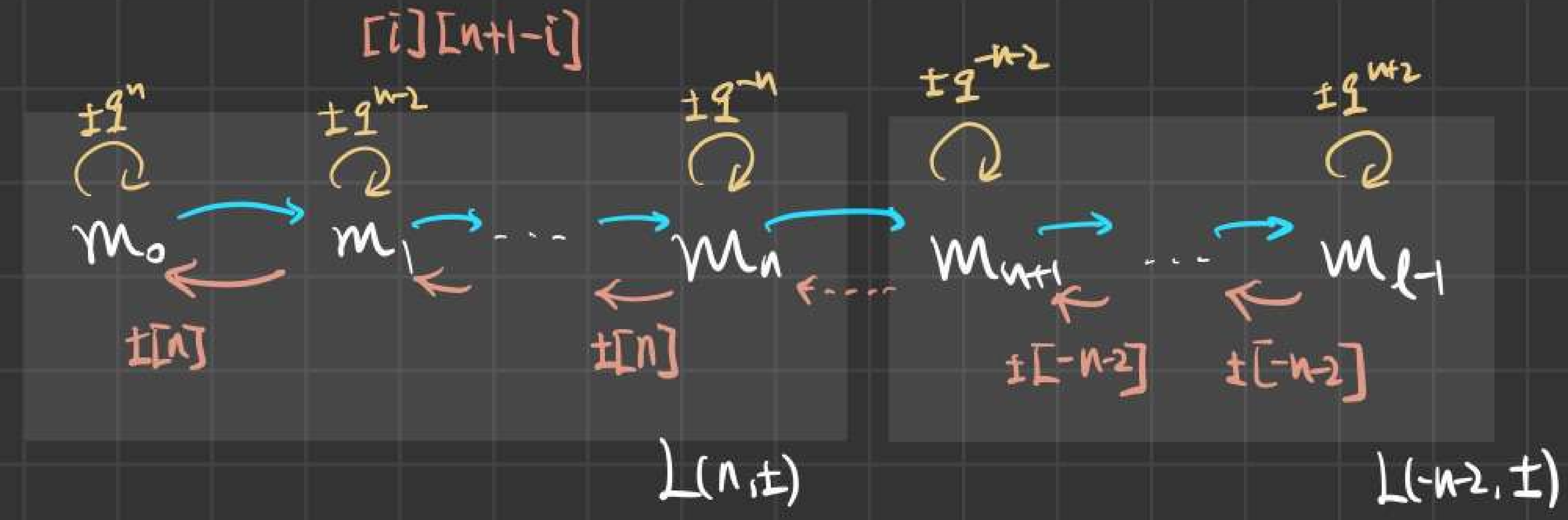
④ M is a semisimple $\mathcal{U}$-mod. (Need $\mathbb{C}$ here)

2. If q is a l-th primitive root of unity with l odd and $l \geq 3$.





$Z_0(\pm q^n)$
$n \in [0, l-1)$



$$0 \to L(-n-2, \pm) \to Z_0(\pm q^n) \to L(n,\pm) \to 0 \quad \text{NOT Split SES!}$$

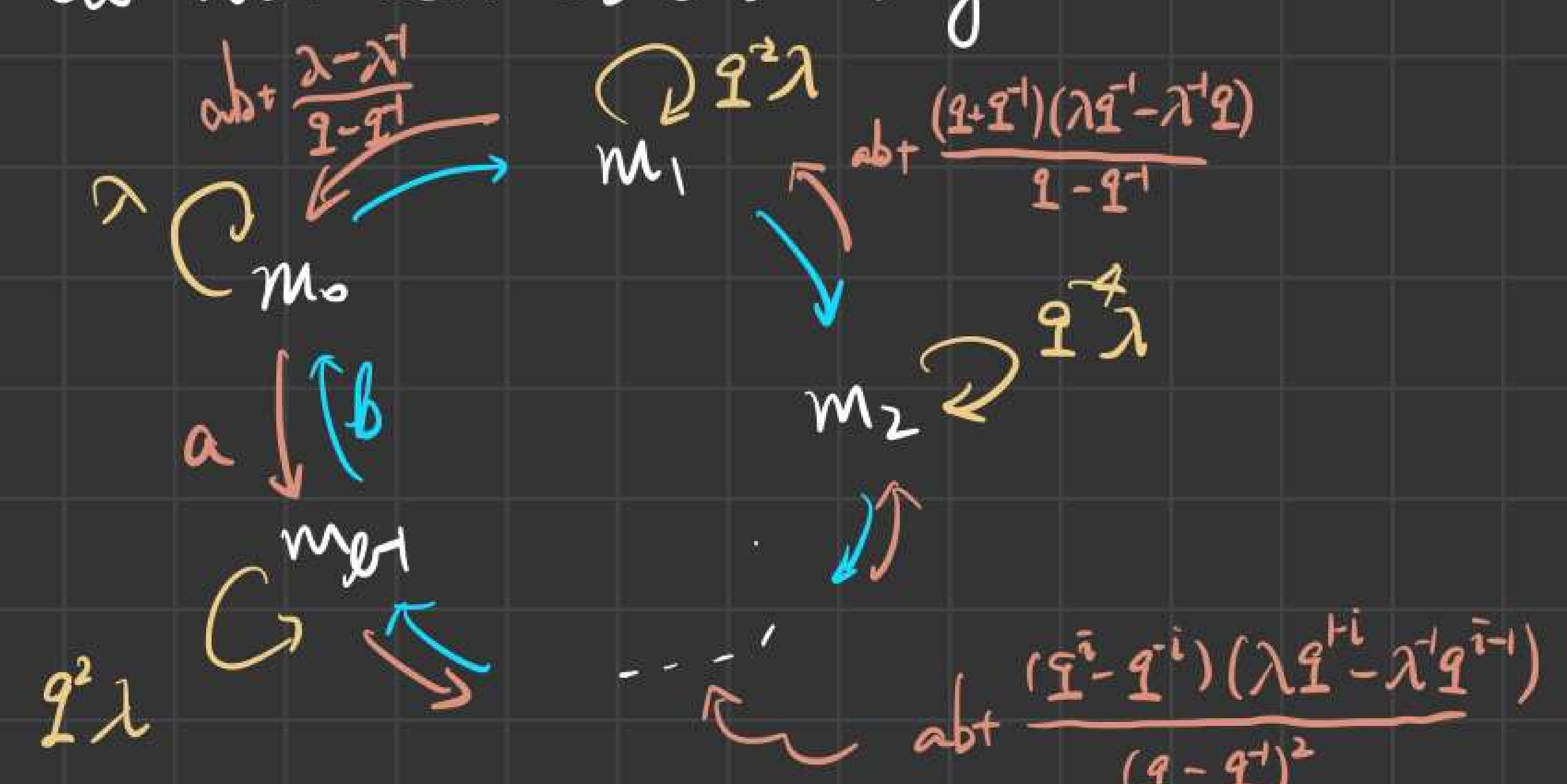
If k is algebraically closed, M is (fin. dim) simple U-mod.

① E^l acts as 0 on M, then $M \simeq Z_0(\lambda)$ or $L(n,\pm)$

② F^l acts as 0 but E^l does NOT, then $M \simeq {}^{\omega}Z_0(\lambda)$
where ω is the involution.

③ E^l & F^l do NOT act as 0. say $E^l \to a$, $F^l \to b$.

M:



Centers of $U_q(sl_2)$:

$$C = FE + \frac{Kq + K^{-1}q^{-1}}{(q-q^{-1})^2} = EF + \frac{Kq^{-1} + K^{-1}q}{(q-q^{-1})^2}$$

1. q is not a root of unity, $Z(U) = k[C]$, $\left(\gamma_{-1} \circ \pi : Z(U) \xrightarrow{\sim} (U^0)^S \right)$
2. q is a primitive l-th root of unity with l odd and $l \geq 3$, $Z(U) = \langle E^l, F^l, K^{\pm l}, C \rangle$

Hopf Algebra Structure of $U_q(sl_2)$

Review: Definition of Hopf algebras:

A Hopf algebra is a vector space A over k equipped with the following linear maps:

$m: A\otimes A \to A$ multiplication

$\iota: k \to A$ unit

$\Delta: A \to A\otimes A$ comultiplication

$\varepsilon: A \to k$ counit

such that

$m\circ(m\otimes id) = m\circ(id\otimes m)$

$m\circ(\iota\otimes id) = id = m\circ(id\otimes\iota)$

$(\Delta\otimes id)\circ\Delta = (id\otimes\Delta)\circ\Delta$

$(\varepsilon\otimes id)\circ\Delta = id = (id\otimes\varepsilon)\circ\Delta$

algebras   coalgebras

and Δ, ε are algebra homomorphisms. m, ι are coalgebra homomorphisms

bialgebras.

and there exists a linear map $S: A\to A$ (antipode) s.t.

$$m\circ(S\otimes id)\circ\Delta = \iota\circ\varepsilon = m\circ(id\otimes S)\circ\Delta.$$

Equivalently, the following diagrams commute.

$$\begin{tikzcd}
A\otimes A\otimes A \arrow[r, "m\otimes id"] \arrow[d, "id\otimes m"'] & A\otimes A \arrow[d, "m"] \\
A\otimes A \arrow[r, "m"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
A\otimes A\otimes A & A\otimes A \arrow[l, "\Delta\otimes id"'] \\
A\otimes A \arrow[u, "id\otimes\Delta"] & A \arrow[l, "\Delta"] \arrow[u, "\Delta"']
\end{tikzcd}$$

$$\begin{tikzcd}
k\otimes A \arrow[r, "\iota\otimes id"] \arrow[dr, "id"'] & A\otimes A \arrow[d, "m"] & A\otimes k \arrow[l, "id\otimes\iota"'] \arrow[dl, "id"] \\
& A &
\end{tikzcd}
\qquad
\begin{tikzcd}
k\otimes A \arrow[dr, "id"'] & A\otimes A \arrow[l, "\varepsilon\otimes id"'] \arrow[r, "id\otimes\varepsilon"] & A\otimes k \\
& A \arrow[u, "\Delta"] \arrow[ur, "id"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
A\otimes A \arrow[d, "id\otimes S"'] & A \arrow[l, "\Delta"'] \arrow[r, "\Delta"] \arrow[d, "\varepsilon"] & A\otimes A \arrow[d, "S\otimes id"] \\
A\otimes A \arrow[r, "m"'] & k \arrow[d, "\iota"] & A\otimes A \arrow[dl, "m"] \\
& A &
\end{tikzcd}$$

Proposition of Hopf algs: 1. antipode is uniquely determined by the conditions.

2. S is automatically a alg antihomo and coalg antihomo with

$$\varepsilon\circ S=\varepsilon \quad , \quad \Delta\circ S = P\circ(S\otimes S)\circ\Delta \quad , \quad P \text{ is exchange map on } A\otimes A.$$

In general, the Hopf structure of A is NOT unique:

If φ is an automorphism or anti-, then we can define a new Hopf structure $({}^{\varphi}\Delta, {}^{\varphi}\varepsilon, {}^{\varphi}S)$ where

$$^{\varphi}\Delta = (\varphi\otimes\varphi)\circ\Delta\circ\varphi^{-1} \quad , \quad {}^{\varphi}\varepsilon = \varepsilon\circ\varphi^{-1},$$

$$^{\varphi}S = \begin{cases} \varphi\circ S\circ\varphi^{-1} & \text{if } \varphi \text{ is an auto} \\ \varphi\circ S^{-1}\circ\varphi^{-1} & \text{if } \varphi \text{ is an antiauto} \end{cases}$$

$U_q(sl_2)$ is a Hopf algebra by

$$\Delta:\ \begin{aligned} E &\to E\otimes 1 + K\otimes E \\ F &\to F\otimes K^{-1} + 1\otimes F \\ K &\to K\otimes K \end{aligned} \qquad \varepsilon:\ \begin{aligned} E &\to 0 \\ F &\to 0 \\ K &\to 1 \end{aligned} \qquad S:\ \begin{aligned} E &\to -K^{-1}E \\ F &\to -FK \\ K &\to K^{-1} \end{aligned}$$

Some formulas:

$$\Delta(K^n) = K^n\otimes K^n \qquad S(K^n) = K^{-n}$$

$$\Delta(E^r) = \sum_{i=0}^{r} q^{i(r-i)} \begin{bmatrix} r \\ i \end{bmatrix} E^{r-i}K^i\otimes E^i \qquad S(E^r) = (-1)^r q^{r(r-1)} K^{-r}E^r$$

$$\Delta(F^r) = \sum_{i=0}^{r} q^{i(r-i)} \begin{bmatrix} r \\ i \end{bmatrix} F^i\otimes F^{r-i}K^{-i} \qquad S(F^r) = (-1)^r q^{-r(r-1)} F^rK^r$$

$$S^2(u) = K^{-1}uK \quad \text{for all } u\in U$$

Rep of tensors:

① $M^* := \mathrm{Hom}_k(M,k)$ dual space: $(uf)(m) = f(S(u)m)$, $\forall u\in U, m\in M, f\in M^*$.

$\varphi': M\to M^{**}$; $\varphi'(m)(f) = f(K^{-1}m)$ a homo of U-mods.

(M and M^{**}, in general, are NOT isomorphism. For fin dim, it is true.)

$M^*\otimes M\to k$; $f\otimes m\mapsto f(m)$ is a homo of U-mods. But the analogous of $M\otimes M^*\to k$ is NOT a homo in general. But we can compose with φ':

$$M \otimes M^* \xrightarrow{\varphi \otimes \mathrm{id}} M^{**} \otimes M^* \to k$$
$$m \otimes f \longmapsto f(K^{-1}m)$$
is a homom of $\mathcal{U}$-mods. NOT the only way.

② $\mathrm{Hom}_k(M,N)$: $u\varphi = \sum u_{(1)} \varphi \circ u_{(2)}$. ($\mathrm{Hom}_k(M,N)$ is a $\mathcal{U}\otimes\mathcal{U}$-mod)

In general, $N \otimes M^* \to \mathrm{Hom}_k(M,N)$: $n \otimes f \mapsto \varphi_{f,n} : m \mapsto f(m) n$ is a $(\mathcal{U}\otimes\mathcal{U})$-homom, but NOT an isomorphism. (For fin dim M, N, it's true)

$$\mathrm{Hom}_{\mathcal{U}}(M,N) = \mathrm{Hom}_k(M,N)^{\mathcal{U}} = \{\varphi \in \mathrm{Hom}_k(M,N) : u\varphi = \varepsilon(u)\varphi\}$$

③ $\mathrm{Hom}_{\mathcal{U}}(M, \mathrm{Hom}_k(N,V)) \xrightarrow{\simeq} \mathrm{Hom}_{\mathcal{U}}(M \otimes N, V)$

④ $\mathrm{tr}_q : \mathrm{End}_k(M) \to k$; $\varphi \mapsto \mathrm{tr}(\varphi \circ K^{-1})$ is a $\mathcal{U}$-homom (quantum trace)

$M \otimes M' \simeq M' \otimes M$

An observation gives: If M & M' fin dim, then $M \otimes M' \cong M' \otimes M$ (as $\mathcal{U}$-mods) since their weight spaces have the same dimension. But P is NOT a $\mathcal{U}$-homom.

Therefore, our goal is to find a functorial isomorphism:

Given $M \xrightarrow{g} N$, $M' \xrightarrow{g'} N'$, the isomorphism R makes the diagram commute

$$\begin{array}{ccc} M \otimes M' & \xrightarrow{R_{M,M'}} & M' \otimes M \\ g\otimes g' \downarrow & & \downarrow g' \otimes g \\ N \otimes N' & \xrightarrow{R_{N,N'}} & N' \otimes N \end{array}$$

Now we find some necessary condition: Take $M = M' = \mathcal{U}$, then set

$$R = R_{\mathcal{U},\mathcal{U}}(1 \otimes 1)$$

$\forall m \in M$, $m' \in M'$, consider the map $\mathcal{U} \to M : a \mapsto am$, $\mathcal{U} \to M' : b \mapsto bm'$

By the functoriality, $R_{M,M'}(m \otimes m') = R \cdot (m' \otimes m)$.

And because $R_{\mathcal{U},\mathcal{U}}$ is a $\mathcal{U}$-homom, $R^{-1} \circ \Delta(u) \circ R = P \circ \Delta(u)$

Now our goal is to find a invertible $R \in U \otimes U$ satisfying $R^{-1} \circ \Delta(u) \circ R = P \circ \Delta(u)$.

- Drinfeld has discovered (in $\mathbb{C}$)

$$R = \left(\sum_{n=0}^{\infty} \frac{(1-q^2)^n}{[n]!} q^{-\frac{n(n-1)}{2}} F^n \otimes E^n \right) \exp\left(\frac{h}{4} H \otimes H\right)$$

where $q = \exp(-\frac{h}{2})$, $K = \exp(-\frac{hH}{2})$. But this $R \notin U$.

Now we consider another construction. (q is NOT a root of unity, char $k \neq 2$)

Step 1. Set $\Theta_n = a_n F^n \otimes E^n \in U \otimes U$, $a_n = (-1)^n q^{\frac{n(n-1)}{2}} \frac{(q-q^{-1})^n}{[n]!}$ $\quad (\Theta_{-1} = 0)$

Then $\Theta_0 = 1 \otimes 1$, $\Theta_1 = -(q-q^{-1}) F \otimes E$. and $a_n = -q^{-(n-1)} \frac{q-q^{-1}}{[n]} a_{n-1}$

and $\Theta = \sum_{n \geq 0} \Theta_n$ is unipotent (bijective), but NOT a U-homom.

with the formula $\quad \Delta(u) \circ \Theta = \Theta \circ {}^{T}\Delta(u) \quad \forall u \in U$.

Step 2. Set $\hat{\Lambda} = \{\pm q^a \mid a \in \mathbb{Z}\}$ weight lattice.

$f : \hat{\Lambda} \times \tilde{\Lambda} \to k^\times$ is a map satisfying $f(\lambda, \mu) = \lambda f(\lambda, \mu q^2) = \mu f(\lambda q^2, \mu)$

$\tilde{f} : M \otimes M' \to M \otimes M'$; $m \otimes m' \mapsto f(\lambda, \mu)\, m \otimes m \quad \forall m \in M_\lambda, m' \in M'_\mu$

(This $\tilde{f}$ exists but NOT unique)

Then $\quad \Delta(u) \circ \Theta^f = \Theta^f \circ (P \circ \Delta)(u)$, where $\Theta^f := \Theta \circ \tilde{f}$.

Theorem: The map $\Theta^f \circ P : M' \otimes M \to M \otimes M'$ is an isomorphism of U-mods. and satisfies the functorial condition.

Quantum Yang-Baxter Equation:

$$R_{12} \otimes R_{13} \otimes R_{23} = R_{23} \otimes R_{13} \otimes R_{12} \quad \text{in } \mathrm{End}_k(V \otimes V \otimes V). \quad \dim V < \infty.$$

In special case ($M = M' = M''$), Θ^f is a solution of the quantum Yang-Baxter equation.

Theorem: $\Theta_{12}^{\mathcal{J}}\,\Theta_{13}^{\mathcal{J}}\,\Theta_{23}^{\mathcal{J}} = \Theta_{23}^{\mathcal{J}}\,\Theta_{13}^{\mathcal{J}}\,\Theta_{12}^{\mathcal{J}}$ in $\mathrm{End}_k(M\otimes M'\otimes M'')$

Pf: ① $LHS = \Theta_{12}\,\tilde{\mathcal{J}}_{12}\,\Theta_{13}\,\tilde{\mathcal{J}}_{13}\,\Theta_{23}\,\tilde{\mathcal{J}}_{23}$

$RHS = \Theta_{23}\,\tilde{\mathcal{J}}_{23}\,\Theta_{13}\,\tilde{\mathcal{J}}_{13}\,\Theta_{12}\,\tilde{\mathcal{J}}_{12}$

When we calculate $\tilde{\mathcal{J}}_{12}\Theta_{13}$ & $\tilde{\mathcal{J}}_{23}\Theta_{13}$, Θ' & Θ'' are involved.

$\tilde{\mathcal{J}}_{12}\circ\Theta_{13} = \Theta'\tilde{\mathcal{J}}_{12}$, $\tilde{\mathcal{J}}_{23}\circ\Theta_{13} = \Theta''\tilde{\mathcal{J}}_{23}$.

where $\Theta' = \sum_{n\geq 0}\Theta'_n = \sum a_n F^n\otimes K^n\otimes E^n$, $\Theta'' = \sum_{n\geq 0}\Theta''_n = \sum a_n F^n\otimes K^{-n}\otimes E^n$

② $LHS = \Theta_{12}\,\Theta'\,\tilde{\mathcal{J}}_{12}\tilde{\mathcal{J}}_{13}\,\Theta_{23}\,\tilde{\mathcal{J}}_{23}$

$RHS = \Theta_{23}\,\Theta''\,\tilde{\mathcal{J}}_{23}\tilde{\mathcal{J}}_{13}\,\Theta_{12}\,\tilde{\mathcal{J}}_{12}$

We find that $\tilde{\mathcal{J}}_{12}\circ\tilde{\mathcal{J}}_{13}$ & Θ_{23} and $\tilde{\mathcal{J}}_{23}\tilde{\mathcal{J}}_{13}$ & Θ_{12} commute.

③ $LHS = \Theta_{12}\,\Theta'\,\Theta_{23}\,\tilde{\mathcal{J}}_{12}\tilde{\mathcal{J}}_{13}\tilde{\mathcal{J}}_{23}$

$RHS = \Theta_{23}\,\Theta''\,\Theta_{12}\,\tilde{\mathcal{J}}_{23}\tilde{\mathcal{J}}_{13}\tilde{\mathcal{J}}_{12}$

Since $\tilde{\mathcal{J}}_{ij}$ are commutative, we only need to consider the first three terms:

$$\begin{aligned}
\Theta_{12}\,\Theta'\,\Theta_{23} &= \sum_{n\geq 0}\sum_{i=0}^{n}\Theta_{12}\,\Theta'_i\circ(1\otimes\Theta_{n-i}) \\
&= \sum_{n\geq 0}\Theta_{12}\circ({}^{T}\!\Delta\otimes 1)(\Theta_n) \\
&= \sum_{n\geq 0}(\Delta\otimes 1)(\Theta_n)\circ\Theta_{12} \\
&= \sum_{n\geq 0}\sum_{i=0}^{n}(1\otimes\Theta_{n-i})\circ\Theta''_i\circ\Theta_{12} \\
&= \Theta_{23}\circ\Theta''\circ\Theta_{12}
\end{aligned}$$

Hexagon Identities:

In the following diagram, R denote maps constructed using suitable $\Theta^{\mathcal{J}}\circ P$

Theorem: Let M, M', M'' fin dim U mod, J satisfies

$$J(\lambda, \mu\nu) = J(\lambda,\mu) J(\lambda,\nu) \quad \& \quad J(\lambda\mu, \nu) = J(\lambda,\nu) J(\mu,\nu)$$

for all weights λ, μ, ν. The the following diagrams commute

$$\begin{tikzcd}
 & M\otimes(M''\otimes M') \arrow[r, "can"] & (M\otimes M'')\otimes M' \arrow[dr, "R"] & \\
M\otimes(M'\otimes M'') \arrow[ur, "R"] \arrow[dr, "can"'] & & & (M''\otimes M)\otimes M' \\
 & (M\otimes M')\otimes M'' \arrow[r, "R"] & M''\otimes(M\otimes M') \arrow[ur, "can"'] &
\end{tikzcd}$$

Hexagon identities

and

$$\begin{tikzcd}
 & (M'\otimes M)\otimes M'' \arrow[r, "can"] & M'\otimes(M\otimes M'') \arrow[dr, "R"] & \\
(M\otimes M')\otimes M'' \arrow[ur, "R"] \arrow[dr, "can"'] & & & M'\otimes(M''\otimes M) \\
 & M\otimes(M'\otimes M'') \arrow[r, "R"] & (M'\otimes M'')\otimes M \arrow[ur, "can"'] &
\end{tikzcd}$$

The proof of this theorem is plain.

Now we consider the existence of J:

A necessary condition is that if all weights are q^a, $a\in\mathbb{Z}$, then

$$J(q^a, q^b) \text{ must be } (q^{\frac{1}{2}})^{-ab}, \ \forall a,b\in\mathbb{Z} \qquad (\text{if } q^{\frac{1}{2}}\in k)$$

Thus, all fin dim U-mod of type 1 (weights $\in \{q^a, a\in\mathbb{Z}\}$) satisfies hexagon identities.

The Quantized Enveloping Algebra $U_q(\mathfrak{g})$

Settings: $\mathfrak{g}$ semisimple Lie alg $/k$

char $k \neq 2$ and char $k \neq 3$ if $\mathfrak{g}$ has component of type G_2.

Φ root system $\quad$ ϖ_α fundamental weight

Π basis of Φ $\quad$ Λ weight lattice

$$a_{\alpha\beta} = \frac{2(\alpha,\beta)}{(\alpha,\alpha)}, \quad d_\alpha = \frac{(\alpha,\alpha)}{2}, \quad \langle \lambda, \alpha^\vee \rangle = \frac{2(\lambda,\alpha)}{(\alpha,\alpha)}$$

Then $U(\mathfrak{g})$ has a presentation with generators $x_\alpha, y_\alpha, h_\alpha$, $\alpha\in\Pi$, and relations

$$[h_\alpha, h_\beta] = 0, \quad [x_\alpha, y_\beta] = \delta_{\alpha\beta} h_\alpha, \quad [h_\alpha, x_\beta] = a_{\alpha\beta} x_\beta, \quad [h_\alpha, y_\beta] = -a_{\alpha\beta} y_\beta$$

and for all $\alpha \neq \beta$,

$$(\operatorname{ad} x_\alpha)^{1-a_{\alpha\beta}}(x_\beta) = 0, \quad (\operatorname{ad} y_\alpha)^{1-a_{\alpha\beta}}(y_\beta) = 0.$$

that is,

$$\sum_{i=0}^{1-a_{\alpha\beta}} (-1)^i \binom{1-a_{\alpha\beta}}{i} x_\alpha^{1-a_{\alpha\beta}-i} x_\beta x_\alpha^i = 0, \quad \sum_{i=0}^{1-a_{\alpha\beta}} (-1)^i \binom{1-a_{\alpha\beta}}{i} y_\alpha^{1-a_{\alpha\beta}-i} y_\beta y_\alpha^i = 0$$

Fix an element $q \in k$, $q \neq 0$ and $q^{2d_\alpha} \neq 1$ for all $\alpha \in \Phi$.

Set $q_\alpha = q^{d_\alpha}$, $\quad [a]_\alpha = [a]_{v=q_\alpha} = \dfrac{q_\alpha^a - q_\alpha^{-a}}{q_\alpha - q_\alpha^{-1}} = \dfrac{q^{ad_\alpha} - q^{-ad_\alpha}}{q^{d_\alpha} - q^{-d_\alpha}}$

$[n]_\alpha^! = [1]_\alpha [2]_\alpha \cdots [n]_\alpha$, $\quad \begin{bmatrix} a \\ n \end{bmatrix}_\alpha = \dfrac{[a]_\alpha^!}{[a-n]_\alpha^! \, [n]^!}$

Definition of $U_q(\mathfrak{g})$:

The quantized enveloping algebra $U_q(\mathfrak{g})$ is a k-alg generated by $E_\alpha, F_\alpha, K_\alpha, K_\alpha^{-1}$ with relations (for all $\alpha, \beta \in \Pi$)

$$K_\alpha K_\alpha^{-1} = 1 = K_\alpha^{-1} K_\alpha \quad , K_\alpha K_\beta = K_\beta K_\alpha$$

$$K_\alpha E_\beta K_\alpha^{-1} = q^{(\alpha,\beta)} E_\beta \quad , \quad K_\alpha F_\beta K_\alpha^{-1} = -q^{(\alpha,\beta)} F_\beta$$

$$E_\alpha F_\beta - F_\beta E_\alpha = \delta_{\alpha\beta} \frac{K_\alpha - K_\alpha^{-1}}{q_\alpha - q_\alpha^{-1}} \qquad \widetilde{U}_q(\mathfrak{g})$$

and for $\alpha \neq \beta$

$$\sum_{i=0}^{1-a_{\alpha\beta}} (-1)^i \begin{bmatrix} 1-a_{\alpha\beta} \\ i \end{bmatrix}_\alpha E_\alpha^{1-a_{\alpha\beta}-i} E_\beta E_\alpha^i = 0$$

$$\sum_{i=0}^{1-a_{\alpha\beta}} (-1)^i \begin{bmatrix} 1-a_{\alpha\beta} \\ i \end{bmatrix}_\alpha F_\alpha^{1-a_{\alpha\beta}-i} F_\beta F_\alpha^i = 0$$

- $K_\lambda = \prod_{\beta\in\Pi} K_\beta^{m_\beta}$, where $\lambda = \sum_{\beta\in\Pi} m_\beta \beta \in \mathbb{Z}\Phi$.
- $U_{q_\alpha}(\mathfrak{sl}_2) \to U_q(\mathfrak{g})$ for each $\alpha\in\Pi$. (The end of this chap will give the injectivity)
- Take a subgroup Γ s.t. $\mathbb{Z}\Phi \subset \Gamma \subset \{\lambda\in\mathbb{Q}\Phi : (\lambda,\beta)\in\mathbb{Z}, \forall\beta\in\Phi\}$

  Then define $U_q(\mathfrak{g},\Gamma) = \langle E_\alpha, F_\alpha, K_\lambda : \alpha\in\Pi, \lambda\in\Gamma \mid \text{relations} \rangle$

  relations: $\quad K_0 = 1, \qquad K_\lambda K_\mu = K_{\lambda+\mu}$,

  $$K_\lambda E_\beta K_\lambda^{-1} = q^{(\lambda,\beta)} E_\beta, \qquad K_\lambda F_\beta K_\lambda^{-1} = -q^{(\lambda,\beta)} F_\beta$$

  $$E_\alpha F_\beta - F_\beta E_\alpha = \delta_{\alpha\beta} \frac{K_\alpha - K_\alpha^{-1}}{q_\alpha - q_\alpha^{-1}}, \qquad \text{and Serre Relations.}$$

  In particular, $U_q(\mathfrak{g}, \mathbb{Z}\Phi) \cong U_q(\mathfrak{g})$

- $U_q(\mathfrak{g})$ is graded: $\deg E_\alpha = \alpha$, $\deg F_\alpha = -\alpha$, $\deg U^0 = 0$.

  Then $K_\lambda u K_\lambda^{-1} = q^{(\lambda,\mu)} u$, $\forall u \in U_\mu$ (Similar for $\widetilde{U}$)

- $U_q(\mathfrak{g})$ is a Hopf alg. define (Δ, ε, S) on each piece of $U_{q_\alpha}(\mathfrak{sl}_2) =: U^\alpha$, $\alpha\in\Pi$

$$\Delta(E_\alpha) = E_\alpha\otimes 1 + K_\alpha\otimes E_\alpha \qquad \varepsilon(E_\alpha) = 0 \qquad S(E_\alpha) = -K_\alpha E_\alpha$$

$$\Delta(F_\alpha) = F_\alpha\otimes K_\alpha^{-1} + 1\otimes F_\alpha \qquad \varepsilon(F_\alpha) = 0 \qquad S(F_\alpha) = -F_\alpha K_\alpha^{-1}$$

$$\Delta(K_\alpha) = K_\alpha\otimes K_\alpha \qquad \varepsilon(K_\alpha) = 1 \qquad S(K_\alpha) = K_\alpha^{-1} \qquad \text{(Similar for } \widetilde{U}\text{)}$$

- $S^2(u) = K_{2\rho}^{-1} u K_{2\rho}$ for all $u \in U$

Basis of $\hat{U}$

Step 1. Construct reps on $M_{k'}$: basis (v_I : I fin seq of simple roots)

Denote $c = (c_\beta)_{\beta \in |\Pi|} \in k'^{|\Pi|}$ nonzero k' tuple. Then $M_{k'}$ has a $\tilde{U}$-mod structure:

$\hat{M}_{k'}(c)$:

$$F_\alpha \cdot v_I = v_{(\alpha, I)}, \quad K_\alpha \cdot v_I = c_\alpha q^{-(\alpha, \mathrm{wt}\, I)} v_I, \quad E_\alpha v_I = \sum_{\substack{1 \le j \le r \\ \beta_j = \alpha}} \frac{c_\alpha q^{-(\alpha, \mu_j)} - c_\alpha^{-1} q^{(\alpha, \mu_j)}}{q_\alpha - q_\alpha^{-1}} v_{(\beta_1, \dots, \hat{\beta_i}, \dots, \beta_r)}$$

where $\mu_j = \sum_{i=j+1}^{r} \beta_i$. (Only (R4) is interesting:

$$E_\alpha F_\alpha \cdot v_I = E_\alpha v_{(\alpha, I)} = \frac{c_\alpha q^{-(\alpha, \mathrm{wt}\, I)} - c_\alpha^{-1} q^{(\alpha, \mathrm{wt}\, I)}}{q_\alpha - q_\alpha^{-1}} v_I + F_\alpha E_\alpha v_I = \frac{K_\alpha - K_\alpha^{-1}}{q_\alpha - q_\alpha^{-1}} v_I + F_\alpha E_\alpha v_I \quad)$$

$\tilde{M}'_{k}(c)$:

$$E_\alpha v_I = v_{(\alpha, I)}, \quad K_\alpha v_I = c_\alpha q^{(\alpha, \mathrm{wt}\, I)} v_I, \quad F_\alpha \cdot v_I = \sum_{\substack{1 \le j \le r \\ \beta_j = \alpha}} \frac{c_\alpha q^{(\alpha, \mu_j)} - c_\alpha^{-1} q^{-(\alpha, \mu_j)}}{q_\alpha - q_\alpha^{-1}} v_{(\beta_1, \dots, \hat{\beta_i}, \dots, \beta_r)}$$

where $\mu_j = \sum_{i=j+1}^{r} \beta_i$.

Step 2. A computational lemma. (proved by induction)

Let I be a seq. There are $c^I_{A,B} \in \mathbb{Z}[v, v^{-1}]$ indexed by fin seqs of simple roots A & B with $\mathrm{wt}\, I = \mathrm{wt}\, A + \mathrm{wt}\, B$ s.t. in U and in $\hat{U}$.

$$\Delta(E_I) = \sum_{A,B} c^I_{A,B}(q) E_A K_{\mathrm{wt} B} \otimes E_B$$

$$\Delta(F_I) = \sum_{A,B} c^I_{A,B}(q^{-1}) F_A \otimes K_{\mathrm{wt} A}^{-1} F_B$$

Cor 1. $\forall \mu \in \mathbb{Z}\Phi, \mu \ge 0$. $\Delta(U^+_\mu) \subset \bigoplus_{0 \le \nu \le \mu} U^+_{\mu-\nu} K_\nu \otimes U^+_\nu$, $\Delta(U^-_{-\mu}) \subset \bigoplus_{0 \le \nu \le \mu} U^-_{-\nu} \otimes K_\nu^{-1} U^-_{-(\mu-\nu)}$

Cor 2. If $x \in U^+_\mu$ and $y \in U^-_{-\mu}$, $\mu = \sum_{\alpha \in \Pi} m_\alpha \alpha$, then

$$S(x) = (-1)^{\mathrm{ht}(\mu)} q^{m(\mu)} K_\mu^{-1} \tau(x) \quad \text{and} \quad S(y) = (-1)^{\mathrm{ht}(\mu)} q^{-m(\mu)} \tau(y) K_\mu$$

where $\mathrm{ht}(\mu) = \sum_{\alpha \in \Pi} m_\alpha$, $m(\mu) = \frac{1}{2}\left((\mu, \mu) - \sum_{\alpha \in \Pi} m_\alpha (\alpha, \alpha)\right)$

Step 3. Basis of $\hat{U}$

Theorem: The elements $F_I K_\mu E_J$ with $\mu \in \mathbb{Z}\Phi$, I, J fin. seq of simple roots are a basis of $\hat{U}$.

Pf. Let V be the subspace of $\hat{U}$ spanned by those elts.

It is easy to show from the calculation that $\hat{U} V \subset V$,

Thus $\hat{U} = V$, i.e. these elts span $\hat{U}$.

Suppose $\sum_{I,J,\mu} a_{I,J,\mu} F_I K_\mu E_J = 0$ in $\hat{U}$ with almost but not all $a_{I,J,\mu} = 0$.

Take I_0 be the sequence such that $\exists\, a_{I_0,J,\mu} \neq 0$ and $\mathrm{wt}\, I_0$ is maximal.

Consider the tensor product $\hat{M}_{k'(C)} \otimes_{k'} \hat{M}'_{k'(C)}$ as a $\hat{U} \otimes k'$-mod. Then

① $$\sum_{I,J,\mu} a_{I,J,\mu} F_I K_\mu E_J (v_\phi \otimes v_\phi) = \sum_{I,J,\mu} a_{I,J,\mu} F_I K_\mu \sum_{A,B} C^J_{A,B}(q) \left(E_A K_{\mathrm{wt} B} v_\phi \otimes v_B \right) = 0$$

Note that $E_A K_{\mathrm{wt} B} v_\phi = 0$ except $A = \phi$. Thus

② $$\sum a_{I,J,\mu} F_I K_\mu (v_\phi \otimes v_J) = \sum a_{I,J,\mu} c^\mu q^{(\mu, \mathrm{wt} J)} F_I (v_\phi \otimes v_J) = 0$$

where $c^\mu = \prod_{\alpha \in \Pi} c_\alpha^{m(\alpha)}$, $\mu = \sum_{\alpha \in \Pi} m(\alpha)\, \alpha$.

Using the lemma again,

③ $$\sum a_{I,J,\mu} c^\mu q^{(\mu, \mathrm{wt} J)} \sum_{C,D} C^I_{C,D}(q^{-1})\, v_C \otimes K^{-1}_{\mathrm{wt} C} F_D v_J = 0$$

The only term in the subspace $v_{I_0} \otimes \hat{M}'_{k'(C)}$ is

④ $$\sum_{J,\mu} a_{I_0,J,\mu} c^\mu q^{(\mu, \mathrm{wt} J)} C^{I_0}_{I_0,\phi}(q^{-1})\, v_{I_0} \otimes K^{-1}_{\mathrm{wt} I_0} v_J$$

$$= \sum_{J,\mu} a_{I_0,J,\mu} c^{\mu - \mathrm{wt} I_0} q^{(\mu, \mathrm{wt} J) - (\mathrm{wt} I_0, \mathrm{wt} J)} \cdot v_{I_0} \otimes v_J = 0$$

Since $v_{I_0} \otimes v_J$ are linearly independent,

⑤ $$\sum_\mu a_{I_0,J,\mu} q^{(\mu, \mathrm{wt} J)} c^\mu = 0, \quad \forall J$$

View it as a polynomial over k in $|\Pi|$ determinates and their inverses.

Since this polynomial vanishes at all $|\Pi|$-tuples (c_α), $a_{I_0, J, \mu} = 0 \ \forall \mu, J$.
This contradicts the choice of I_0. Thus, they are linearly independent.

Triangular Decomposition of $\mathcal{U}$

Denote $\mathcal{U}^+_{\alpha\beta}$ and $\mathcal{U}^-_{\alpha\beta}$ the Serre relation in $\hat{\mathcal{U}}$, i.e.

$$\mathcal{U}^+_{\alpha\beta} = \sum_{i=0}^{1-a_{\alpha\beta}} (-1)^i \begin{bmatrix} 1-a_{\alpha\beta} \\ i \end{bmatrix}_\alpha E_\alpha^{1-a_{\alpha\beta}-i} E_\beta E_\alpha^i$$

$$\mathcal{U}^-_{\alpha\beta} = \sum_{i=0}^{1-a_{\alpha\beta}} (-1)^i \begin{bmatrix} 1-a_{\alpha\beta} \\ i \end{bmatrix}_\alpha F_\alpha^{1-a_{\alpha\beta}-i} F_\beta F_\alpha^i$$

and $I^\pm$ the ideal in $\hat{\mathcal{U}}^\pm$ generated by $\mathcal{U}^\pm_{\alpha\beta}$.

Prop: The two-sided ideal generated by $\mathcal{U}^\pm_{\alpha\beta}$ in $\hat{\mathcal{U}}$ is equal to the image of $\hat{\mathcal{U}}^- \otimes \hat{\mathcal{U}}^0 \otimes I^+$ (resp. $I^- \otimes \hat{\mathcal{U}}^0 \otimes \hat{\mathcal{U}}^+$) under the multiplication, say $V^\pm$.

Pf. Only show for $\mathcal{U}^+_{\alpha\beta}$.

V^+, as a vector space, is spanned by $u\, \mathcal{U}^+_{\alpha\beta} E_I$, $u \in \hat{\mathcal{U}}$, I suitable fin seq.
Thus V^+ is a left ideal.
By a direct but complicated calculation, V^+ is a two-sided ideal.
Thus, $V^+ = \langle \mathcal{U}^+_{\alpha\beta} : \alpha \neq \beta \rangle_{\hat{\mathcal{U}}}$

Thm: The multiplication map $m: \mathcal{U}^- \otimes \mathcal{U}^0 \otimes \mathcal{U}^+ \longrightarrow \mathcal{U}$ is an isom. of vector spaces.
Pf. Let I be the kernel of $\pi: \hat{\mathcal{U}} \twoheadrightarrow \mathcal{U}$, i.e. two-sided ideal generated by $\mathcal{U}^\pm_{\alpha\beta}$.
It is obvious that $I \cap \hat{\mathcal{U}}^0 = 0$. Thus $m: \hat{\mathcal{U}}^0 \xrightarrow{\sim} \mathcal{U}^0$
And $I \cap \hat{\mathcal{U}}^\pm = I^\pm \Rightarrow \hat{\mathcal{U}}^\pm / I^\pm \xrightarrow{\sim} \mathcal{U}^\pm$

$$\hat{\mathcal{U}}/I \cong \hat{\mathcal{U}}^- \otimes \hat{\mathcal{U}}^0 \otimes \hat{\mathcal{U}}^+ / (\hat{\mathcal{U}}^- \otimes \hat{\mathcal{U}}^0 \otimes I^+ + I^- \otimes \hat{\mathcal{U}}^0 \otimes \hat{\mathcal{U}}^+) \cong \hat{\mathcal{U}}^-/I^- \otimes \hat{\mathcal{U}}^0 \otimes \hat{\mathcal{U}}^+/I^+ \cong \mathcal{U}^- \otimes \mathcal{U}^0 \otimes \mathcal{U}^+$$

Rmk. $\mathcal{U}^\pm$ is isom to the algebra generated by $E_\alpha (F_\alpha)$, $\alpha \in \Pi$ and relation $\mathcal{U}^\pm_{\alpha\beta}$
(commutative)

Compared with Lie algebras, PBW Thm of $U_q(\mathfrak{g})$ is quite hard. The reasons are:

① The graded associative alg of $\mathcal{U}$ is NOT commutative

② The root vectors of $\mathcal{U}$ is NOT clear.

Thus, we can not prove the PBW now.

But by analyzing the grading of $\mathcal{U}^{\pm}$, we can get $F_\alpha^{r_\alpha} K_\mu^{t_\mu} E_\beta^{s_\beta}$ are linearly independent. ($I^{\pm}_{\pm r\alpha} = 0 \quad \forall r \in \mathbb{Z}^{>0}$)

Thus, $U_q(sl_2) \longrightarrow U_q(\mathfrak{g})$ is an imbedding.